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Abstract
Representations of concepts from multivariate statistics are studied using multilinear algebra. Applications are given to the analysis of seasonal multivariate time series.

The first paper concerns a class of permutation matrices which are representations of permutation operators on tensor spaces.

In the second paper, the derivation of moments and cumulants of Hilbert space valued random variables are studied. Relations between moments and cumulants - as vector space elements - are given which generalize well-known results for scalar random variables. Representations of differentials are discussed and results for differentials of matrix valued functions with matrix arguments are derived. Moments and cumulants of the non-central Wishart distribution are given.

In the third paper, seasonal multivariate time series are investigated using the concepts described in the first two papers. Estimators of unknown parameters are given and their asymptotic distributions are derived. Test statistics for testing homogeneity in time and independence of subprocesses are given and also their asymptotic distributions.

Key words Permutation operators, moment, cumulant, matrix derivative, multivariate autoregressive process, periodically varying parameters, parameter estimation, model verification

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Björn Holmquist

The thesis consists of this summary and the following three papers:

[A] Holmquist, B. (1985a): The direct product permuting matrices. TFMS-3036. Dept. of Mathematical Statistics, University of Lund. (Published in Linear and Multilinear Algebra 17, (1985) 117-141.)

[B] Holmquist, B. (1985b): Moments and cumulants from generating functions of Hilbert space-valued random variables and an application to the Wishart distribution. TFMS-3037. Dept. of Mathematical Statistics, University of Lund.

[C] Holmquist, B. (1985c): Innovation correlated multivariate autoregressive time series with periodically correlated parameters. TFMS-3038. Dept. of Mathematical Statistics, University of Lund.

1. INTRODUCTION

Multivariate statistical models provide useful techniques for analysing many problems. Similarly as for univariate statistical models, differentiation in statistics is strongly connected with the development of techniques such as maximum likelihood estimation, derivation of moments and cumulants from generating functions, derivation of Fisher information matrices etc.

There are two different ways for performing the analysis, namely the coordinate based (algebraic) or basis dependent approach and the coordinate free (geometric) or vector space approach.

Both alternatives have their advantages and disadvantages. Although coordinate free considerations are convenient for analysis and understanding, representations in coordinate spaces are almost necessary for computer aided calculations.

The price for the insight gained in using only geometrical concepts in a coordinate free approach is the considerable work needed for restating the results in a coordinate based setup when implementing the results on a computer.

On the other hand, the algebraic approach gives the results in a coordinate representation suitable for implementation. However, the derivation of the result may be reduced to algebraic manipulations in which the geometry of the problem and result may be hidden.

When using the algebraic approach to differentiation, easily handled representations of differentials of mappings of matrix arguments are of great use.

Differentiation of scalar or matrix valued functions of matrix arguments is a task which requires simple rules for an efficient evaluation. Examples of functions with matrix arguments occurring in statistics are likelihood functions with vector and matrix arguments, densities and moment generating functions of vector and matrix random variables and covariance matrices with patterned structures. Differentiation of these functions is a tool often used for maximisation and minimisation of these functions or for obtaining moments of random variables.

In later years, articles and books have dealt with the notational aspects of the topic of differentiation of matrices with respect to matrices; see for instance MacRae (1974), McDonald and Swaminathan (1973) and Rogers (1980).

We use a special matrix representation of the differential mapping of a matrix function with matrix argument which give easily evaluated rules for calculating differentials.

Quite often the elements of the argument matrix are dependent. An example in statistics is afforded by the maximization of a likelihood with respect

to a covariance matrix. Differentiation of the likelihood with respect this covariance matrix needs a theory for differentiation with respect to symmetric matrices.

The notational aspects and rules for differentials with respect to symmetric matrices and more general dependencies between the elements have been dealt with in some articles; cf. for example McCulloch (1982) and Nel (1980); see also Rogers (1980).

Our approach is to employ projection operators to project the representation obtained without these restriction onto the space spanned by the dependent elements.

A special class of projection operators is the symmetrizers on tensor spaces.

DEFINITION Let X be a linear space and $X^{\langle r\rangle}$ the r'th tensor power of X. The symmetrizer S_{1_r} on $X^{\langle r\rangle}$ is the linear mapping

$$(1) \qquad S_{1_r} = \sum_{\sigma \in S_{1_r}} P_r^{\otimes}(\sigma)/r!$$

where $P_r^{\otimes}(\sigma)$ is the permutation operator on $X^{\langle r\rangle}$ accociated with σ, i.e.

$$P_r^{\otimes}(\sigma)(x_1 \otimes \ldots \otimes x_r) = x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(r)}$$

where $x_i \in X$, $i=1,\ldots,r$, and S_{1_r} is the symmetric group of degree r. □

For $r=2$, the symmetrizer on the space of m-square matrices projects any m-square matrix A onto its symmetric part $A_+ = \frac{1}{2}(A + A^t)$. This symmetrizer is connected with differentials of mappings with respect to symmetric matrices and with second order moments and cumulants of random variables.

Representations of symmetrizers are strongly connected with representations of mappings differentiated with respect to higher order symmetric vectors and with representations of higher order moments and cumulants.

Coordinate space representations of symmetrizers are obtained by replacing the permutation operators involved in (1) by their images under a homomorphic mapping into the general linear group of non-singular matrices.

The thesis can roughly be divided into two parts; the first part, papers [A] and [B], concerns representations of certain projection operators, differential mappings, moments and cumulants. Paper [A] deals with a class of permutation matrices, called the direct product permuting matrices, which in a special case are coordinate representations of the permutation operators $P_r^{\otimes}(\sigma)$, $\sigma \in S_{1_r}$ on tensor spaces. Paper [B] treats representations of moments and cumulants of Hilbert space-valued random variables and relations to representations of

differentials of generating functions. We study general representations of differentials of functions with arguments represented in coordinate spaces. The moments and cumulants in matrix formulation of the non-central Wishart distribution are derived as an example of the relations between representations of differentials and moments and cumulants and of the applicability of the rules for the representation of the differential.

The second part, paper [C], concerns the analysis of a non-stationary time series model showing the periodic behaviour occuring in many real data time series, as for example in monthly river flow data. In the analysis given in an algebraic approach, concepts described in the first two papers are used in order to get a neat notation. We study estimation procedures, model verification, and methods for choosing simpler submodels of the original model, by testing for time homogeneity and independence of subprocesses.

2. PAPER A: DIRECT PRODUCT PERMUTING MATRICES

The commutation matrix is known under many names, see Henderson and Searle (1981) for a review of names, notations, properties and definitions. It occurs in many situations in multivariate statistical analysis, e.g. in differentials of quadratic forms of a variable such as the log likelihood of a normally distributed random vector variable, or in expressions of the moments of multivariate random variables (Magnus and Neudecker (1979)).

The properties of the commutation matrix, $T_{m,n}$, are usually investigated on their own. However, with $I_{m,m} = I_m \otimes I_m = I_{m^2}$,

$$S_{\frac{1}{2}}^{(m)} = \tfrac{1}{2}(I_{m,m} + T_{m,m})$$

is a matrix representation of the projection operator $S_{\frac{1}{2}}$, with domain on the space of vectorized m-square matrices, and range in the space of vectorized symmetric m-square matrices; that is, $A = A_+ + A_- \in R^{m,m}$ where $A_+ = \tfrac{1}{2}(A + A^t)$ is symmetric and $A_- = \tfrac{1}{2}(A - A^t)$ is skew-symmetric, is mapped to A_+ by

$$S_{\frac{1}{2}}^{(m)} \mathrm{vec}(A) = \mathrm{vec}(A_+)$$

Higher order symmetrizers are connected with higher order moments and cumulants and differentials of mapping with respect to higher order symmetric vectors.

In the paper we investigate the direct product permuting matrices (DPPM), with properties similar to those of the commutation matrix. One of the main properties of these matrices is that they permute the order of a Kronecker

product of several matrices, when the DPPM's corresponding to the permutation and its inverse are pre- and postmultiplied respectively to the Kronecker product. We derive these matrices as images of a special product of identity matrices. Other properties of the DPPM's are given, and the trace and determinant of the matrices are derived, generalizing known results for the commutation matrix.

3. PAPER B: MATRIX REPRESENTATIONS OF DIFFERENTIALS

A coordinate space representation of differentials is proposed and several rules are given for differentiation of coordinate space mappings with respect to arguments realized in coordinate spaces such as column vectors and matrices.

The merits of the representation given are seen when differentials of inverse matrices, transposed matrices and products of matrices are to be calculated and when the chain-rule for the representation is applied in a manner essentially not more complicated than in the scalar case.

Rules for differentiation of mappings expressed in linearly dependent representatives are also given.

4. PAPER B: MULTIVARIATE MOMENTS AND CUMULANTS

Moments of multivariate random variables are usually defined as product-moments, that is, as expectations of products of elements of the random vector (see e.g. Kendall and Stuart (1976)). Such an element-oriented definition requires a large number of indices for labelling the moments and depends on the specific representation of the random variables.

Cumulants of multivariate random variables are usually defined in a similar manner as product-cumulants, that is, as partial derivatives at zero of the logarithm of the moment generating function. Like the moments these product-cumulants are identified by a large number of indices.

The relation between a certain product-cumulant and product-moments of various multi-indices requires quite complicated rules for calculating the coefficient for a product-moment of a certain multi-index in the linear relationship between product-cumulants and product-moments (see e.g. Kendall and Stuart (1976)).

One reason for dealing with moments and cumulants as elements of vector spaces rather than dealing with coordinate concepts is the geometrical in-

sight gained by the coordinate free approach. Another reason is the notational aspect; regarding a specific moment or cumulant rather than its elements (coordinates) requires one index instead of a set of multi-indices for the elements.

The geometrical approach is used in the first part of the paper to define higher order moments of random variables in Hilbert spaces. It is shown how moments and cumulants of Hilbert space-valued random variables can be obtained from generating functions. By differentiation of the moment generating function and regarding the dual of the differential, well-known results for univariate random variables are generalized to vector-valued random variables.

We also derive expressions for moments and cumulants of polynomial transformations of vector-valued variables, expressed in terms of the moments and cumulants of the original variables.

Different coordinate space representations of moments and cumulants are discussed in the paper. The first order moment, $E(X)$, is usually and naturally defined as the coordinate-wise expectations of the elements. For column vector representations X of random variables the second order central moment is usually represented as the dispersion matrix $\mathcal{D}(X) = E(XX^t) - E(X)E(X^t)$. For matrix valued random variables a possible representation of the second order moment is obtained by converting the matrix into a column vector by means of the vec operator and using the representation used for column vectors. Higher order moments of matrices (or column vectors) are not easily represented by using this approach.

The representation of moments used in the paper is

$$\mu_r = E(X^{\langle r \rangle})$$

for the r'th moment about zero, where $X^{\langle r \rangle} = \otimes_{k=1}^{r} X$ and $\otimes$ is the Kronecker product. This realization is applicable for random variables represented as column vectors or matrices as well as to other arbitrary representations of random variables as soon as the direct product of these representations is defined.

In the second part of the paper, the coordinate free relations in the first part are rewritten in terms of basis dependent relations in coordinate spaces. For this purpose we use the coordinate space representations of the projection operators onto completely symmetric spaces over a coordinate space which are obtained from the coordinate space representations of permutation operators given in paper [A].

In the third part of the paper, the differentiation technique is applied to the moment generating function of a non-central Wishart distributed matrix for obtaining its moments and cumulants in matrix notation.

5. PAPER C: A SEASONAL AUTOREGRESSIVE TIME SERIES

Autoregressive time series models with periodically time varying parameters are used in hydrology to generate sequences of river flow like monthly data.

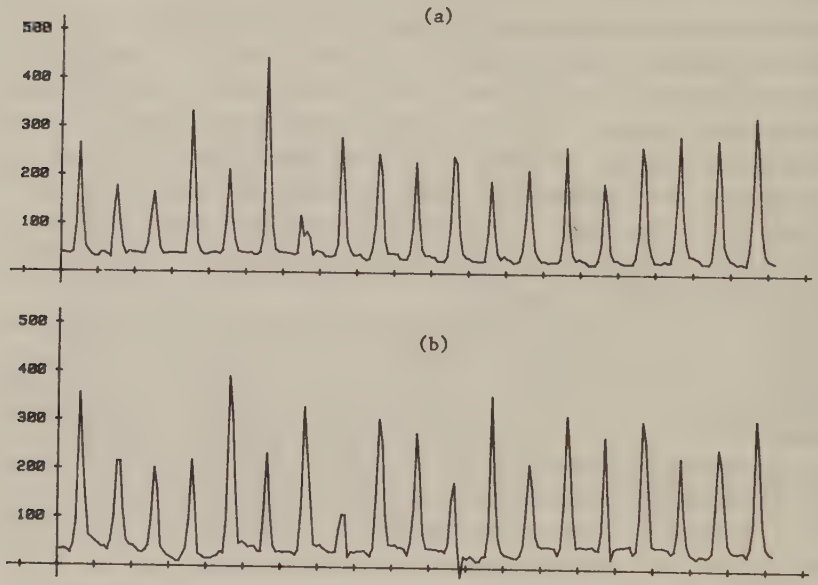

Figure River flow data.
(a) Monthly data (m^3/sec.) from 19 consecutive years from River Kunduz, Afghanistan.
(b) Simulated data from a first order autoregressive process of the type studied in the paper.

The motivation for the approach for modelling this type of time series is that the underlying mechanism, which is represented by the dependence backwards in time, undergoes changes under different seasons of the year. This change in the mechanism may not only concern the mean level but also the influence of previous levels of the process and the amount of randomness in the generating process at different time points.

The approach of periodic autoregressive time series has an advantage over the integrated autoregressive time series when the interpretation of the dependence of previous levels is more easily understood than the dependence on previous differences — one period apart — of levels.

A multivariate generalization of a process studied by Jones and Brelsford (1967), Pagano (1978) and Troutman (1979) is studied in the paper:

$$(2) \qquad y_k = \sum_{i=1}^{p_k} \Phi_i(k) y_{k-i} + e_k$$

where $\Phi_i(k)=\Phi_i(k+d)$, $p_k=p_{k+d}$ and $E(e_k e_k^T)=\Sigma(k)=\Sigma(k+d)$ for some positive integer d.

Emphasis of the investigation is laid on the estimation of unknown parameters, model checking and methods for reducing the initial model to simpler models by investigating the homogeneity in time of the parameters, and the interrelation between subprocesses.

A generalization of a technique used by Pagano (1978) for the univariate case is employed for rewriting the process as a multivariate stationary autoregressive process, where the observable variables in the new process are subsequent periods of the old observable variables. The technique allows the theory for stationary (multivariate) autoregressive processes to be applied, and results for the autoregressive time series with periodically correlated parameters can be derived.

Two different approaches for estimation of the parameters are investigated. One approach is to regard observations corresponding to parameters at a particular time point within the period. Another approach is to estimate the parameters in the deperiodized process from which the estimates of the parameters in the original process are derived. The deperiodization technique has been used to derive estimates of the parameters in the original process by means of matrix factorization technique and the corresponding factorization of distributions. The asymptotic distributions for long realizations are given for the derived estimators.

Portmanteau statistics are proposed for testing the consistency of data with the model.

The multivariate process allows modelling of interelation between several subprocesses. We consider estimation of parameters in two special cases of the general model (2). One of these submodels is a multivariate and periodic version of a process studied by Risager (1981) where the interrelation between the subprocesses is through the innovation process $\{e_k\}$. The other special case is the model in which the innovation processes for the corresponding subprocesses are uncorrelated. These models describe two basically different types of origin of the dependence.

We suggest statistics for testing submodels of the general model. The asymptotic distributions of the statistics are given when the length of the observed realization increases.

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